On Optimizing GNSS Multi-Frequency Carrier Phase Combinations for Precise Positioning

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Abstract

Traditional combinations of dual-frequency GPS carrier phase data have focused on three combinations, namely the widelane (WL), the narrowlane (NL), and the ionospheric-free (IF) combination. All of these combinations suffer from limitations, which affect the overall accuracy of precise positioning. For example, the ambiguities resolved by using the IF combination are not integer. Although resolving ambiguities using the WL combination is easier than resolving ambiguities of the component frequencies, the resulting positioning accuracy is degraded somewhat from the inherent amplification of the noise. Furthermore, previous studies of phase combinations have regarded errors on both frequencies as independent. In reality, errors such as tropospheric and ionospheric delays are correlated between frequencies. Thus, combinations of phase measurements may actually have reduced error magnitudes than the original measurements. The focus of this paper is to study various combinations of multi-frequency carrier phase observations. The goal is to improve our final positioning accuracy. High accuracy positioning requires both correct ambiguity resolution and the minimization of the combined errors. The approach followed in this paper, regards the combination “derived measurement” as a transformation of an observation space composed of the two original measurements. Since errors such as tropospheric delay and multipath are *physically* independent, the error properties of the combination can be represented as a summation of the transformations of individual error sources. Thus, transformations can be created that reduce multipath, ionospheric error, or any individual or combination of error sources. Furthermore, since it is known that the original phase measurements include integer ambiguities, combinations can be derived that preserve the integer nature of the ambiguities. Case studies are presented which optimize positioning accuracy on short to medium length baselines.

1 Introduction

It is well known that the achievable positioning accuracy from GPS is affected by various error sources, including atmospheric, satellite orbit, multipath and noise. An effective method for alleviating the error sources is through mathematical modelling, whereby the correlations between receivers, satellites and frequencies are taken into account. Linear carrier phase combinations can be used to reduce the errors correlated between frequencies (consider for example the ionospheric-free combination, Klobuchar, 1996). The purpose of this paper is to develop a linear combination with minimum error to allow for precise positioning. The parameters used in the linear combination are dependent on the nature of the error sources affecting the carrier phase observations. Some advantages that are inherent when using a combination of phases as opposed to the L1 and L2 phase information is that there is a reduction in the data transmission requirements (i.e. for RTK applications or dense engineering networks). In addition, accurate knowledge of the variance-covariance properties of the error sources are required in the adjustment of observations used to obtain the final position values.

Our discussion begins with an overview of common linear carrier phase combinations and their properties. This is followed by the derivation of an ‘optimal’ linear phase combination. The context in which ‘optimal’ is referred to will also be discussed in detail. In order to test the validity and performance of the newly derived ‘optimal’ phase combination a number of numerical simulation tests were conducted. The results of these tests are also presented and analyzed. Finally, some conclusions are drawn based on the numerical tests conducted and suggestions for future work are provided.

2 Linear Phase Combinations

Given carrier phases measured from two receivers and two satellites, the *double-differenced* phase combination for each frequency can be written as

$$\nabla\Delta\phi_{L1} - \frac{\nabla\Delta R}{\lambda_{L1}} - \nabla\Delta N_{L1} = \frac{\nabla\Delta T + \nabla\Delta\delta r}{\lambda_{L1}} + \lambda_{L1}\nabla\Delta I + \nabla\Delta n_{L1} + \nabla\Delta m_{L1} \quad (1a)$$

$$\nabla\Delta\phi_{L2} - \frac{\nabla\Delta R}{\lambda_{L2}} - \nabla\Delta N_{L2} = \frac{\nabla\Delta T + \nabla\Delta\delta r}{\lambda_{L2}} + \lambda_{L2}\nabla\Delta I + \nabla\Delta n_{L2} + \nabla\Delta m_{L2} \quad (1b)$$

where $\nabla\Delta$ is the double-difference operator, ϕ_{L1} and ϕ_{L2} are the phases measured on the L1 and L2 frequencies in cycles, λ_{L1} and λ_{L2} are the corresponding carrier phase wavelengths, $\nabla\Delta R$ is the double-differenced range between receivers and satellites and $\nabla\Delta N$ is the ambiguity term. $\nabla\Delta T$, $\nabla\Delta\delta r$, $\nabla\Delta I$, $\nabla\Delta n$ and $\nabla\Delta m$ are the double-differenced tropospheric, orbital, ionospheric, noise and multipath errors, respectively. Furthermore, the variance-covariance matrix for the double-difference pair can be written as

$$\nabla\Delta\mathbf{C}_l = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad (2)$$

where the diagonal elements are given by

$$c_{11} = \frac{\nabla\Delta\sigma_T^2 + \nabla\Delta\sigma_{\delta r}^2}{\lambda_{L1}^2} + \lambda_{L1}^2\nabla\Delta\sigma_I^2 + \nabla\Delta\sigma_n^2 + \nabla\Delta\sigma_m^2$$

$$c_{22} = \frac{\nabla\Delta\sigma_T^2 + \nabla\Delta\sigma_{\delta r}^2}{\lambda_{L2}^2} + \lambda_{L2}^2\nabla\Delta\sigma_I^2 + \nabla\Delta\sigma_n^2 + \nabla\Delta\sigma_m^2$$

and the cross-covariance terms are given by

$$c_{12} = c_{21} = \frac{\nabla\Delta\sigma_T^2 + \nabla\Delta\sigma_{\delta r}^2}{\lambda_{L1}\cdot\lambda_{L2}} + \lambda_{L1}\lambda_{L2}\nabla\Delta\sigma_I^2$$

where the variance of the tropospheric and orbital errors are in m^2, the noise and multipath variances are in $cycles^2$ and the ionospheric variance is in units of TEC/m^2. Thus, the two double-differences shown in Eq. (1a) and (1b) are correlated due to the tropospheric, ionospheric and orbital errors affecting both phases. Note that the noise and multipath errors are assumed uncorrelated between frequencies, which is valid only in the case that the receiver does not use L1-aiding to track the L2 signal (Kaplan, 1996). This assumption is maintained throughout this paper.

A linear phase combination, ϕ_*, of two double-differences can be expressed as

$$\nabla\Delta\phi_* = \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} \nabla\Delta\phi_{L1} \\ \nabla\Delta\phi_{L2} \end{bmatrix} \quad \Big| \quad a, b \in Z \quad (3)$$

where a and b are the coefficients of the linear combination. The coefficients are required to be integer values in order to ensure that the resulting ambiguity of the linear combination is also integer-valued. A more detailed discussion on integer ambiguity resolution techniques which result in maximum positioning accuracy is given in Teunissen (1996). The process of forming the linear combination is essentially that of projecting a two-dimensional "observation space" into a single pseudo-observation. The variance of this pseudo-observation $\nabla\Delta\sigma_*^2$ can be calculated via the propagation of errors and the variance-covariance matrix of the original observations. The final expression is as follows:

$$\begin{aligned}\nabla\Delta\sigma_*^2 = {} & a^2\left(\nabla\Delta\sigma_n^2 + \frac{\nabla\Delta\sigma_T^2}{\lambda_1{}^2} + \lambda_1^2\nabla\Delta\sigma_I^2\right) \\ & + 2ab\left(\frac{\nabla\Delta\sigma_T^2}{\lambda_1\lambda_2} + \lambda_1\lambda_2\nabla\Delta\sigma_I^2\right) \\ & + b^2\left(\nabla\Delta\sigma_n^2 + \frac{\nabla\Delta\sigma_T^2}{\lambda_2{}^2} + \lambda_2^2\nabla\Delta\sigma_I^2\right)\end{aligned} \quad (4)$$

where the variance is given in $cycles^2$. In Eq. (4) the tropospheric and orbital errors have been grouped together and denoted by $\nabla\Delta\sigma_T^2$ and the noise and multipath errors are collectively denoted by $\nabla\Delta\sigma_n^2$. The resulting wavelength of the new pseudo-observation λ_* can be derived in a straight-forward manner and is given as follows (Hoffmann-Wellenhof et al., 1994):

$$\lambda_* = \frac{\lambda_{L1}\cdot\lambda_{L2}}{a\cdot\lambda_{L2} + b\cdot\lambda_{L1}} \quad (5)$$

3 Development of 'Optimal' Linear Phase Combination

The variance-covariance matrix of the estimated parameters $\mathbf{C}_x$ stemming from a least-squares adjustment is given by the well-known formula (Krakiwsky, 1975)

$$\mathbf{C}_x = \left(\mathbf{A}^T (\mathbf{B}\mathbf{C}_l \mathbf{B}^T)^{-1} \mathbf{A}\right)^{-1} \tag{6}$$

where $\mathbf{A}$ is a Jacobian matrix with respect to the unknown parameters, $\mathbf{B}$ is the Jacobian matrix with respect to the observations and $\mathbf{C}_l$ is the variance-covariance matrix of the observations. Further inspection of Eq. (6) shows that the matrices $\mathbf{A}$ and $\mathbf{B}$ are invariant under various choices for the linear combination coefficients, a and b, assuming that the observation equations are expressed in meters. On the other hand, Eq. (4) states that the $\mathbf{C}_l$ matrix *is* directly affected by the choice of these coefficients. Thus, it is possible to change the achievable positioning accuracy through an "optimal" selection of a and b. In the following two subsections, the optimal combination will be discussed for certain special cases followed by a general procedure for determining optimal combinations.

3.1 Optimal Combinations for Specialized Cases

The variance of a linear combination given by Eq. (4) can be broken up into individual error components. For example, the noise/multipath error variance in cycles^2 is given by

$$\nabla\Delta\sigma_*^2(n) = \nabla\Delta\sigma_n^2 (a^2 + b^2) \tag{7}$$

In this case, the pair of coefficients (a,b) that minimize the error variance are (0,1) and (1,0), assuming that the set (0,0) is not admissible. This implies that on very short baselines, where atmospheric effects (almost) cancel, it is advisable to simply use the L1 measurement, rather than a linear combination. Using a similar approach, the tropospheric error variance for the linear combination can be expressed by

$$\nabla\Delta\sigma_*^2(T) = \nabla\Delta\sigma_T^2 \cdot \left(\frac{a^2}{\lambda_1^2} + \frac{2ab}{\lambda_1\lambda_2} + \frac{b^2}{\lambda_2^2}\right) \tag{8}$$

Eq. (8) is composed of a constant, $\nabla\Delta\sigma_T^2$, multiplied by a quadratic form composed of the coefficient values a and b. As a result, it can be shown that the tropospheric variance (in cycles^2) can be eliminated by a linear combination that satisfies the relation $a/b = -\lambda_1/\lambda_2$. A similar analysis for the ionospheric error reveals that the combination that eliminates the ionospheric error is $a/b = -\lambda_2/\lambda_1$, which is the well-known ionospheric-free combination (Klobuchar, 1996).

It is important to realize that a linear combination which minimizes a particular error in units of cycles does not necessarily minimize the error in units of meters. For example, the tropospheric error variance in m^2 is given by

$$\begin{aligned}
\nabla\Delta\sigma_*^2(T) &= \nabla\Delta\sigma_T^2 \cdot \left(\frac{a^2}{\lambda_1^2} + \frac{2ab}{\lambda_1\lambda_2} + \frac{b^2}{\lambda_2^2}\right) \\
&= \nabla\Delta\sigma_T^2 \cdot \left(\frac{a^2}{\lambda_1^2} + \frac{2ab}{\lambda_1\lambda_2} + \frac{b^2}{\lambda_2^2}\right) \cdot \left(\frac{\lambda_{L1}\lambda_{L2}}{a\lambda_{L2} + b\lambda_{L1}}\right)^2 \\
&= \nabla\Delta\sigma_T^2 \cdot \left(\frac{a^2}{\lambda_1^2} + \frac{2ab}{\lambda_1\lambda_2} + \frac{b^2}{\lambda_2^2}\right) \cdot \left(\frac{\lambda_{L1}^2\lambda_{L2}^2}{a^2\lambda_{L2}^2 + 2ab\lambda_{L1}\lambda_{L2} + b\lambda_{L1}^2}\right) \\
&= \nabla\Delta\sigma_T^2
\end{aligned} \tag{9}$$

which shows that it is impossible to reduce the tropospheric error in units of meters by using a linear combination. Similarly, it can be shown that the noise error in meters tends towards zero as a and b approach infinity. Unfortunately, in such a scenario, the wavelength tends to zero, making reliable ambiguity resolution impossible.

3.2 General Procedure for Determining Optimal Combinations

The previous section illustrates one of the key difficulties in defining 'optimality' for linear combinations. While it may be possible to minimize the cyclic error variance of a combination, it is not guaranteed that this combination will minimize the error variance in units of meters. Similarly, while short wavelengths generally reduce the error variance in m^2 (which is desirable for positioning accuracy) it is more difficult to resolve ambiguities reliably under such circumstances. Thus, the definition of optimality is somewhat arbitrary. For the purposes of this study, an optimal combination is one that minimizes the total error variance in meters^2, while maintaining an error variance in cycles^2 that is at least equal to that of the L1 observable. In this way, the positioning accuracy is maximized, but ambiguity resolution does not become more difficult than in the L1-only case.

Figure 1 shows the relationship between the error components considered for various values of *a* and *b*. The noise error variance function forms a paraboloid with global minimum at $(a,b) = (0,0)$. The tropospheric and ionospheric error variance functions form parabolic cylinders, with global minima lying along the lines described in Section 3.1 and shown graphically in Figure 1. The total error variance function is simply the sum of these surfaces and thus forms a saddle-shaped surface, whose *ridge* lies along the line described by

$$\left(\frac{a}{b}\right)^* = \frac{-\frac{\lambda_1}{\lambda_2}\nabla\Delta\sigma_T^2 - \frac{\lambda_2}{\lambda_1}\nabla\Delta\sigma_I^2}{\nabla\Delta\sigma_T^2 + \nabla\Delta\sigma_I^2} \quad (10)$$

To determine an optimal integer combination, integer pairs lying on either side of the line described by Eq. (10) are chosen, and the combinations with total error variances less than that of the L1 double-differenced observations are retained. The remaining combination that minimizes the total error variance in m^2 is then chosen as the optimal candidate.

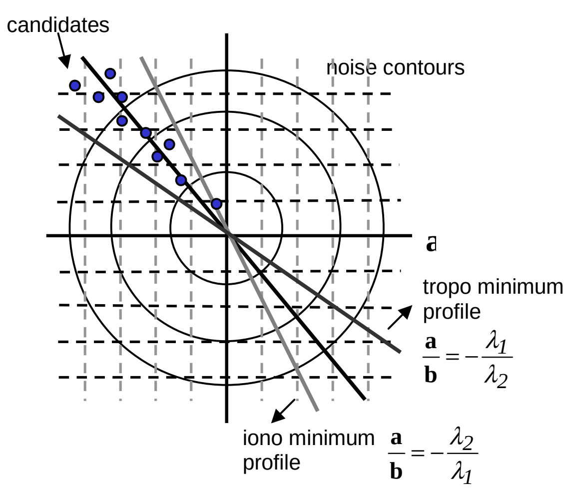

Fig. 1. Selection of an optimal linear combination.

4 Description of Simulated Tests

In order to determine the numerical accuracy obtained from using the 'optimal' combination discussed in the previous sections, a number of simulation tests were conducted. A 20 minute observation span at a 30 seconds observation rate was simulated with seven satellites visible throughout the duration of the test (giving six double-differenced observations per epoch). Dual-frequency observations were assumed with the receiver-to-receiver separation set at 100 km. The test scenarios involved the generation of the theoretical variance-covariance matrix using known models, the derivation of the 'optimal' phase combination (i.e. determining values for the coefficients *a* and *b* in Eq. (3)), and comparing the results for the accuracy of the solved baseline components from various combinations, namely, L1-only, L1/L2, widelane, and ionospheric-free, by inspection of the variance-covariance matrix.

In Table 1, the simulation model parameters are listed. Specifically, the standard deviations and correlation lengths (denoted by *s*) assumed for the double-differenced error models used are listed. In addition, the temporal correlations used for each of the error sources are given. The selection of these model input parameters are site/day specific, for example the ionospheric correlation distance can be expected to shorten under storm conditions. In this study, the noise, tropospheric and ionospheric parameters are *moderate* values collected from studies by Radovanovic (2001), Radovanovic, et al. (2001), Collins and Langley (1999), Klobuchar (1996), Schaer (1999), Raquet (1998) and El-Rabanny (1994). Due to space restrictions, the full details of these error models will not be presented.

Table 1. Parameters for Error Models

Modelled Error	Model input Parameters	Temporal Correlations
Noise multipath	σ = 0.5mm (L1) σ = 0.6mm (L2)	0s (noise) 300s (multipath)
tropospheric (and orbital)	σ = 0.02m (L1/L2 at zenith) *s* = 350 km	1000s
ionospheric	σ = 0.28m (L1 at zenith) *s* = 1500 km	1000s

4.1 Determination of Optimal Combination

Given the error variance models for each of the error sources mentioned above, the integer ambiguity candidates were chosen via the procedure discussed in Section 3.2. Figure 2 shows the candidate combinations, as well as the slope of the ionospheric/tropospheric minimization line. Figure 3 shows the error in cycles^2 (top figure) for each of the integer combination candidates selected. The heavy black line indicates the error in cycles^2 of the L1 observation. From Figure 3 it can be seen that the error of the combination increases as the values for a and b increase. This is primarily due to the amplification of noise as indicated by Eq. (7). According to our 'optimality' criterion (discussed in Section 3.2) the final step for determining the optimal combination is to choose the remaining integer candidate that minimizes the total variance in m^2. The results for each combination candidate are also shown in Figure 3 (bottom figure). Once again, the heavy black line indicates the variance of the L1 observation. Interestingly, as the coefficient value a increases, the total variance converges to the tropospheric variance in m^2 (as shown by Eq. (9)). After performing the above mentioned steps, *the overall 'optimal' integer combination* (a, b) *is given by the pair (4, -3)*. This resulting combination has an associated wavelength of 13.4 cm and an error variance (in m^2) of ~ 5 times smaller than the L1 observable (or roughly twice as small in units of cycles).

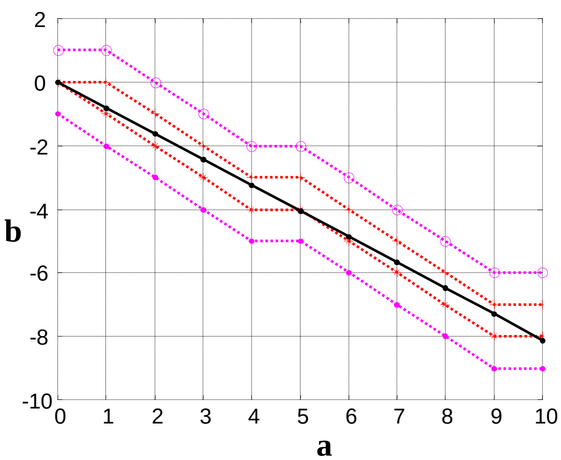

Fig. 2. Candidate coefficient combinations (**b**/**a** = -0.82, λ_1/λ_2 = 0.78, λ_2/λ_1 = 1.28)

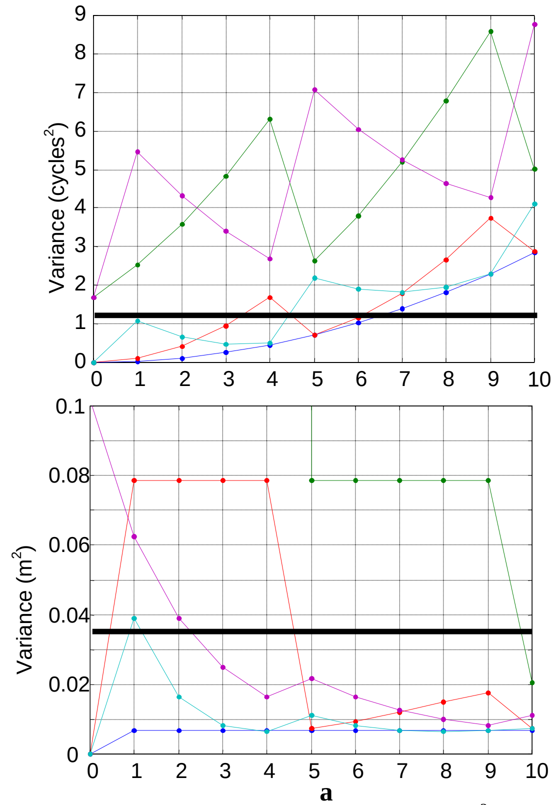

Fig. 3. Candidate combination variances in cycles^2 (top) and m^2 (bottom)

5 Effects of Linear Combinations on Positioning Accuracy - Results

The goal of developing a method to calculate optimal linear combinations is to encapsulate the information contained in dual-frequency observations into a single pseudo-observation with minimal loss of information. As a result, the utility of using the linear combination can be gauged by examining the effects of the combination on achievable positioning accuracies. Using the simulated data described in the previous section the variance-covariance matrix of baseline parameters was generated using several different observational scenarios, namely the dual-frequency case, the L1-only case, and linear combinations corresponding to the widelane, ionospheric-free and the optimal combination calculated in the previous section. In all cases, a *full* observation variance-covariance matrix is generated and rigorously propagated.

The resulting standard deviations are presented in Table 2 for the float case and in Table 3 for the fixed-ambiguity cases. In both cases, the dual-frequency scenario provides the best positioning accuracy. This is understandable since it uses both frequencies and the known correlations between the phases as described by Eq. (2). On the other hand, the L1-only solution provides poorer positioning accuracy since only half of the available information is used. The widelane linear-combination results in the worst accuracy, mainly due to its long wavelength, which causes the combination to suffer from a high variance in m^2 and the amplified noise and ionospheric errors. The ionospheric-free combination provides a positioning accuracy nearly identical to the dual-frequency case.

Table 2. Standard Deviations for Float Solutions (in cm)

	L1	L1/L2	WL	IF	Optimal
N	41	17	56	17	20
E	44	16	59	16	19
H	50	21	68	21	24

Table 3. Standard Deviations for Fixed Solutions (in cm)

	L1	L1/L2	WL	IF	Optimal
N	6.8	2.0	8.5	2.0	2.4
E	6.4	1.7	7.6	1.7	2.1
H	12.9	3.4	16.5	2.4	4.3

This is due to the dominance of the ionospheric error in the simulation, which is about 10 times larger than the tropospheric error. Unfortunately, the ionospheric-free combination does not contain an integer ambiguity and thus is not admissible. The use of the 'optimal' linear combination shows significant improvement in the resulting positioning accuracy as compared to the L1-only and WL cases. In fact, it is only slightly worse than the dual frequency case due to its inability to utilize information contained orthogonal to the principle component of the variance-covariance matrix.

6 Conclusions and Future Work

An approach for developing linear phase combinations for precise positioning has been presented. The parameters used in the linear combination were estimated using the information about the variance-covariance properties of the GPS error sources which affect the achievable position accuracy. In this case, an 'optimal' combination was proposed in the sense that the observation variance *in* m^2 was minimized while maintaining a variance in units of cycles that was at most equal to the cycle variance of the original observations. This implies that the float positional accuracy is balanced with successful ambiguity resolution. From a practical user point of view, this proposed combination provides a synthetic-observation (combination of two existing observations) which minimizes the data transmission requirements by half as compared to using the full L1/L2 observations. In addition, the size of the variance-covariance matrices is significantly reduced allowing for more efficient inversion. Numerical tests conducted to simulate real scenarios show that the positioning accuracy can be improved by a factor of ~2.5 through the use of an optimized linear phase combination as compared to L1-only.

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