

Variance-Covariance Modelling of Carrier Phase Errors for Rigorous Adjustment of Local Area Networks

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Abstract. A method of rigorously combining GPS data collected at several ground stations is presented. The adjustment procedure is based on a simultaneous adjustment of GPS data collected at all observing station and the use of a hybrid variance-covariance model of the phases observed. The study is strictly restricted to the short (< 10 km) baseline scenario commonly encountered in local-area deformation monitoring, and as such, deal primarily with multipath and noise errors.

Theoretical noise and multipath models are developed, and verified using real data collected from short baselines. In addition, a method to determine level of multipath at each station in a local area network is presented. Finally, the temporal correlations of noise and multipath are investigated, revealing a correlation period for multipath in the 200-300s range and following an exponential decay.

The developed models are then used to process a four-station static network of receivers with known coordinates. For comparison purposes, the network is also processed a) using no modelling of covariances, b) only considering only mathematical correlations between double differences, and c) using empirical variance models but no temporal correlation modelling. It is shown that proper variance-covariance modelling has minimal effect on the position estimates returned, but is crucial for correct estimation of the parameter statistics returned by the least-squares adjustment.

Keywords. Variance-Component Estimation, Network Adjustment, Error Modelling, Multipath

1 Introduction

Traditional techniques of high-precision surveying with the Global Positioning System have focused on using double-differences between carrier phase measurements made at two or more receivers to several satellites simultaneously. Due to the

correlated nature of the errors affecting these carrier phases, differencing the phases between satellites and receivers can significantly reduce the residual error affecting a position solution, especially on baselines of short length (< 10 km). On such baselines, the residual error remaining is mainly due to noise and multipath, resulting in residual errors at the centimetre level.

In the subsequent processing of carrier phase differences, the mathematical correlations between double differences are typically modelled correctly (Hoffman-Wellenhof, 1994) However, the original physical correlations between phases are usually neglected in the diagonal variance-covariance (VC) matrix of observations used. In addition, temporal correlations between phases are usually ignored if the adjustment incorporates more than one epoch of data.

Several investigations have studied the effects of mis-modelling the variance-covariance matrix used in GPS processing. Beutler et al (1987) has shown that errors modeling the mathematical correlations of double differences causes millimetre level errors in position estimates. By extension, errors in modeling the physical correlations of the original phases should cause effects at a similar level, a conclusion supported by (Grodecki, 1997).

However, the major effect of neglecting physical correlations of phases (both spatial and temporal) is in the statistics returned by the least-squares adjustment for station positions. This is because the variance-covariance matrix of the solved parameters depends directly on the variance-covariances of the observations and their mathematical correlations. For example, Han and Rizos (1995) verified that statistical estimates are always over-optimistic when correlations between observations are ignored.

El-Rabbany (1994) presented a study of the temporal correlations affecting GPS observations, and concluded that they followed an exponential decay, with a correlation period of 1200 seconds. This study did not consider the effect of variations in baseline length, nor the temporal correlations of

the individual error sources. Nonetheless, it yields an important starting point in the modelling of the temporal correlations, in particular in dealing with the difficulties in inverting the resulting, fully populated variance-covariance matrix.

This study approaches the problem of variance-covariance modelling from a hybrid theoretical-empirical approach. The local-area scenario is studied, where noise and multipath errors dominate. The analysis in general begins with a theoretical consideration of the individual error sources, followed by the development of a model to describe the variance and temporal correlation. Key parameters of the model are then derived from empirical data. An experiment involving a four-receiver network is presented that illustrates the improvement in the statistical estimates achievable when proper VC modelling is undertaken, while confirming the relative insensitivity of the estimates of the parameters themselves to the VC matrix used.

2 Variance-Covariance Modelling of GPS Carrier Phases

The covariance of any two measurements between two receivers (A and B) and two satellites (i and j) can be expressed as

$$\begin{aligned} \mathbf{s}_{ij}^{AB} &= E[(\mathbf{f}_i^A - E(\mathbf{f}_i^A)) \cdot (\mathbf{f}_j^B - E(\mathbf{f}_j^B))] \\ &= E \left[\begin{pmatrix} \frac{1}{I} \left[c \cdot (\mathbf{dt}_{sati}^A - \mathbf{dt}_{recA}^i) + \mathbf{dr}_i^A + \right] \\ \mathbf{dT}_i^A - \mathbf{dT}_j^A + m_i^A + n_i^A \\ \frac{1}{I} \left[c \cdot (\mathbf{dt}_{satj}^B - \mathbf{dt}_{recB}^j) + \mathbf{dr}_j^B + \right] \\ \mathbf{dT}_j^B - \mathbf{dT}_i^B + m_j^B + n_j^B \end{pmatrix} \right] \end{aligned} \quad (1)$$

where \mathbf{f} is the measured fractional phase in cycles, λ is the carrier wavelength, c is the speed of light, \mathbf{dt}_{sat} and \mathbf{dt}_{rec} are the satellite and receiver clock offsets, \mathbf{dr} is the orbital error projected along line-of-sight, T and I are the tropospheric and ionospheric delays, and m_f and n_f are the multipath and noise errors on the carrier.

Note that the expected value of ϕ_i^A is equal to the range and ambiguity of the carrier phases and that the error sources are assumed to have zero mean.

Eq. (1) can be further simplified by assuming that the error sources are physically uncorrelated among themselves. This is valid since each error source stems from a different physical process. For example, there is no reason why the value of the satellite clock offset should at all affect the actual

tropospheric delay contained within a measured carrier phase. As a result, the variance expression becomes :

$$\begin{aligned} \mathbf{s}_{ij}^{AB} &= \frac{1}{I^2} E \left[\begin{pmatrix} c \cdot (\mathbf{dt}_{sati}^A \mathbf{dt}_{satj}^B + \mathbf{dt}_{recA}^i \mathbf{dt}_{recB}^j) + \mathbf{dr}_i^A \mathbf{dr}_j^B + \\ \mathbf{dT}_i^A \mathbf{dT}_j^B - \mathbf{dT}_i^A \mathbf{dT}_j^B + m_i^A m_j^B + n_i^A n_j^B \end{pmatrix} \right] \\ &= \mathbf{s}_{ij}^{AB}(\mathbf{dt}_{sat}) + \mathbf{s}_{ij}^{AB}(\mathbf{dt}_{rec}) + \mathbf{s}_{ij}^{AB}(\mathbf{dr}) + \mathbf{s}_{ij}^{AB}(\mathbf{dT}) \\ &\quad + \mathbf{s}_{ij}^{AB}(\mathbf{dI}) + \mathbf{s}_{ij}^{AB}(m) + \mathbf{s}_{ij}^{AB}(n) \end{aligned} \quad (2)$$

where the covariances of the individual error sources have been introduced. Eq. (2) implies that the covariance of a set of phases can be derived simply by directly adding the covariances of the individual error sources.

Furthermore, since operations such as differencing are linear, the resulting double-differenced variance-covariance matrices can be also decomposed according into individual error components. For example (Radovanovic et al, 2000),

$$\begin{aligned} \mathbf{C}^{DD} &= \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{B}^T \\ &= \mathbf{B} \cdot (\mathbf{C}_n + \mathbf{C}_m + \mathbf{C}_T + \mathbf{C}_I + \mathbf{C}_o) \cdot \mathbf{B}^T \\ &= \mathbf{B} \cdot \mathbf{C}_n \cdot \mathbf{B}^T + \mathbf{B} \cdot \mathbf{C}_m \cdot \mathbf{B}^T + \mathbf{B} \cdot \mathbf{C}_T \cdot \mathbf{B}^T \\ &\quad + \mathbf{B} \cdot \mathbf{C}_I \cdot \mathbf{B}^T + \mathbf{B} \cdot \mathbf{C}_o \cdot \mathbf{B}^T \end{aligned} \quad (3)$$

where \mathbf{C}^{DD} is the total variance-covariance matrix of observations, \mathbf{C}_n , \mathbf{C}_m , \mathbf{C}_T , \mathbf{C}_I and \mathbf{C}_o are the variance-covariance matrices for the noise, multipath, tropospheric, ionospheric and orbital errors, respectively, and \mathbf{B} is the operator matrix corresponding to the double-differencing operation.

In local area networks, the double-differenced variance-covariance matrices for the tropospheric, ionospheric, and orbital errors are significantly smaller in magnitude than those corresponding to noise and multipath. As a result, this study only deals with these latter error sources. As shown in the sequel, the noise and multipath variance-covariances can be separated through an appropriate selection of experiments.

3 Noise Variance-Covariance Modelling

Noise error is an effect that is unavoidable in any measurement process. In the case of GPS, it is the error in phase and code measurements due to imperfect tracking of the GPS signal by the phase and delay lock loops. As a result, it is internal to the receiver and is usually independent between

satellites since separate loops are dedicated to each signal tracked.

Furthermore, measurement noise is typically considered uncorrelated in time beyond the predetection integration period (typically 20 ms). Thus it can be treated as a white noise process, assuming the output phases and ranges are not smoothed internally by the receiver. This is an assumption justified for the L1 carrier phase measurement (Raquet, 1998).

In static applications, the predominant error component in carrier noise is jitter of the phase lock loop caused by thermal noise. This can be expressed as (Kaplan, 1996) :

$$\mathbf{s}_n = \mathbf{s}_{PLL} = \frac{1}{2p} \sqrt{\frac{B_n}{c/n_o} \left(1 + \frac{1}{2Tc/n_o}\right)} \quad (4)$$

where B_n is the carrier loop noise bandwidth, c/n_o is the carrier to noise power expressed as $10^{C/N_{o/10}}$ with C/No in dB-Hz, T is the predetection integration time and λ is the carrier wavelength.

Consider the scenario of three receivers (A,B and C), making simultaneous measurements to three satellites (1, 2 and 3). Nine phases are measured, yielding four unique double differences. Since the noise is uncorrelated between satellites and receivers, the double difference noise variances-covariances can be written as :

$$\begin{aligned} \mathbf{s}^2(\nabla\Delta_{12}^{AB}n) &= \mathbf{s}_1^{A^2}(n) + \mathbf{s}_1^{B^2}(n) + \mathbf{s}_2^{A^2}(n) + \mathbf{s}_2^{B^2}(n) \\ \mathbf{s}^2(\nabla\Delta_{12}^{AB}n, \nabla\Delta_{13}^{AB}n) &= \mathbf{s}_1^{A^2}(n) + \mathbf{s}_1^{B^2}(n) \\ \mathbf{s}^2(\nabla\Delta_{12}^{AB}n, \nabla\Delta_{12}^{AC}n) &= \mathbf{s}_1^{A^2}(n) + \mathbf{s}_2^{A^2}(n) \\ \mathbf{s}^2(\nabla\Delta_{12}^{AB}n, \nabla\Delta_{13}^{AC}n) &= \mathbf{s}_1^{A^2}(n) \end{aligned} \quad (5)$$

Furthermore, since carrier noise is considered white, there is no correlation between epochs. Thus by using Eq. (4), estimates of the signal power received and manufacturer specifications, it is possible to calculate the variance-covariance matrix for the noise component.

An empirical model of the noise profile affecting a given receiver-antenna setup can be determined using a zero-baseline test, in which the signal from one antenna is split and passed to two receivers. Since all errors external to the receivers will be identical in this scenario, and thus cancel upon double differencing, the remaining residuals are due to noise. Since the carrier-to-noise power in Eq. (4) is elevation dependent, noise variance can be expressed as the product of a zenith noise variance,

\mathbf{s}_n^2 multiplied by a mapping function, resulting in the expression (Raquet, 1998):

$$\mathbf{s}^2(\nabla\Delta_{12}^{AB}n) = 2 \cdot (m(\mathbf{e}_1) + m(\mathbf{e}_2)) \cdot \mathbf{s}_n^2 \quad (6)$$

where $m(\mathbf{e}_1), m(\mathbf{e}_2)$ are the mapping functions for the observed satellites and it is assumed that both receivers have identical noise profiles.

Raquet (1998) and Radovanovic et al (2000) confirmed that the noise mapping function can be assumed equal to 1 for elevation angles greater than 40° . Assuming that the base satellite is always above this elevation, the mapping function for other elevation angles can be determined using Eq. (6) where the $\mathbf{s}^2(\nabla\Delta_{12}^{AB}n)$ can be determined for a particular elevation from analysis of the collected double-differenced zero-baseline data and the zenith variance is calculated by analysis of the residuals when both satellites are above 40° elevation.

To compare the theoretical and empirical approaches, noise profiles for a NovaTel OEM3 receiver were generated. A model 502 antenna was used, equipped with a low-loss splitter to deliver the signal to both receivers simultaneously. Signal loss specifications were provided by the manufacturers. In addition, the antenna gain profile for the antenna used was provided. Combined with estimates of the received signal power (GPS-ICD-200, 1999) and atmospheric losses (Spilker, 1996), the theoretic variance of the carrier phase noise at various elevations was calculated using Eq. (4). The theoretical and empirical noise variances are shown in figure 1, which shows close agreement between the two methods. Noise models for other receiver/antenna combinations are presented in Radovanovic et al (2000).

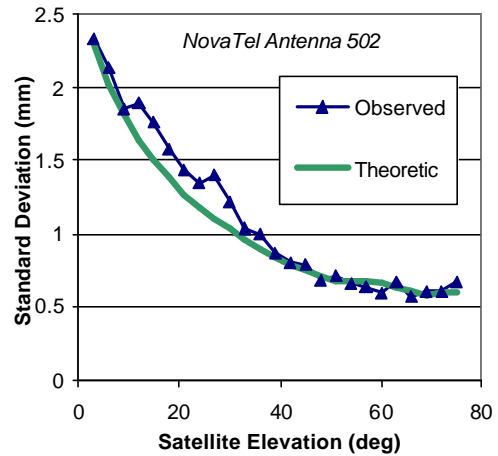


Figure 1. Theoretical and Noise Profiles for NovaTel OEM 3 equipped with #502 Antenna

4 Multipath Variance-Covariance Modelling

The multipath error affecting a carrier phase measurement due to n reflectors is given by (Ray, 2000) :

$$df = a \tan \left[\frac{\sum_i b_i \sin \Delta f_i}{1 + \sum_i b_i \cos \Delta f_i} \right] \quad (7)$$

where b_i is the reflection coefficient of reflected signal i , Df is the phase delay of reflected signal i and A is antenna gain pattern of direct signal.

For the receiver to be able to lock onto the direct signal, the sum of the reflective coefficients must be less than the antenna gain attenuating the direct signal. This implies that $(b_i/A) \ll 1$. Using this assumption, Eq. (7) can be approximated by

$$\begin{aligned} df &\approx \left[\frac{\sum_i b_i \sin \Delta f_i}{1 + \sum_i b_i \cos \Delta f_i} \right] \\ &\approx \left(\sum_i b_i \sin \Delta f_i \right) \cdot \left(1 - \sum_i b_i \cos \Delta f_i \right) \\ &\quad \left[1 + \left(\sum_i b_i \cos \Delta f_i \right)^2 \right] \cdot \left[1 + \left(\sum_i b_i \cos \Delta f_i \right)^4 \right] \end{aligned} \quad (8)$$

The variance of multipath can be derived by expanding the above approximation, and calculating the variance of each term, taking into account the orthogonality of the sinusoids involved. The resulting expression, neglecting higher order terms, is given by

$$\begin{aligned} s^2(m) &= \left[\frac{(b_{mean}^2 + b_{var}) \cdot n \cdot I^2}{2} \right] \cdot \frac{1}{A^2} \\ &= s_m^2 \cdot m(\mathbf{e}) \end{aligned} \quad (9)$$

where b_{mean} is the mean reflectance of reflectors in the environment, b_{var} is the variance of the reflectivities of these reflectors, and n is the number of reflectors in the environment. Note that the terms in the brackets can be considered "site-specific" and constant over the observing period, while the antenna gain varies with the elevation of the direct signal. Thus the multipath variance can be expressed as the product of a zenith multipath

variance multiplied by an elevation dependent mapping function.

Using a similar procedure, it can be shown that multipath is uncorrelated between satellites and receivers. Thus the double-differenced multipath variance-covariance model follows the same form as Eq. (5).

If a local-area network of receivers with known coordinates is observed over time, and the noise profiles of the receivers are known, it is possible to determine the double-differenced multipath variances for each baseline by removing the predicted noise variance from the observed variance, based on the principle of independence of error sources. The remaining double-differenced multipath variances can be further decomposed into the multipath variances for the individual phases by noting that

$$\begin{aligned} s^2(\nabla \Delta_{12}^{AB} m) &= \\ &= (m_A(\mathbf{e}_1) + m_A(\mathbf{e}_2)) \cdot s_{m_A}^2 + (m_B(\mathbf{e}_1) + m_B(\mathbf{e}_2)) \cdot s_{m_B}^2 \end{aligned} \quad (10)$$

where it is assumed that the elevation angles to a satellite are equal at all receivers. The mapping function terms can be calculated from Eq. (9) and the antenna gain profile for the antennas used in the network. Alternatively, they can be empirically determined using the approach employed for the analysis of noise previously presented. The site-dependent zenith multipath variances can then be calculated from a least-squares adjustment of the observed double-differenced multipath variances as observed in the network over various elevation angles.

Such a test was performed using a network consisting of two NovaTel OEM3 receivers and two Trimble 4000SSI receivers, located on the roof of the Engineering Building at the University of Calgary. Baseline lengths were less than 10 metres, ensuring that tropospheric, orbital and ionospheric errors essentially differenced out completely. Two hours of data was collected, resulting in 6000 observed double-differenced residuals. These residuals were binned according to elevation angle and the variance of each bin calculated.

The predicted noise variances were then removed from these observed variances using a noise model previously derived. Finally, zenith multipath variances and mapping functions were derived for each station. These are presented in figure 2. In addition, a theoretic multipath noise profile for the NovaTel receivers was generated from available

antenna specifications and is plotted alongside the empirical results, showing good agreement. In addition, it is evident that the shapes of both NovaTel profiles are similar to each other, as are the shapes of the Trimble curves. This further illustrates the dependence of multipath variance on elevation angle, through the antenna gain pattern.

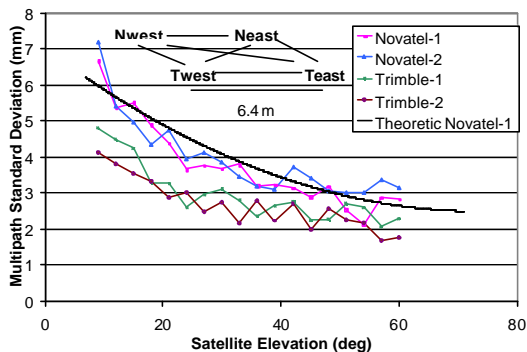


Figure 2. Multipath Profiles for Four Station Network.

The temporal correlation of multipath is modelled as an exponential decay. As explained in El-Rabbany (1994), such a model imparts a structure to the variance-covariance matrix that allows for efficient inversion. Using the residuals observed in the aforementioned network, temporal correlations were derived for various elevation and plotted in figure 3. An exponential decay is evident, with a correlation period of between 200 to 300 seconds. However, no distinct dependence of the correlation period with elevation angle can be noted. In addition, the authors believe that this correlation period is site dependent, since the variation of the multipath depends on the geometry of reflectors in the environment. However, more work is needed to characterise this behaviour.

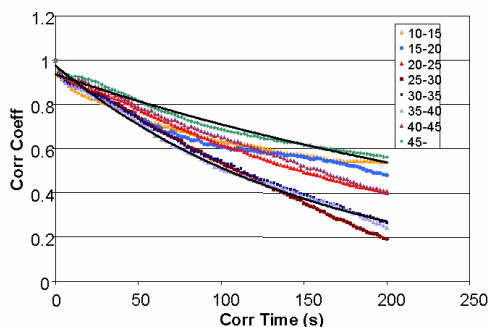


Figure 3. Temporal Correlation of Multipath

5 Effects of Variance-Covariance Modelling on Network Adjustment

The nature of the variance-covariance matrices implicit in GPS observation processing have been discussed in the previous sections. However, most commercial processors do not correctly apply these matrices, mainly due to the extra overhead involved in storing and inverting fully populated matrices. As a result, it becomes imperative to determine what affects incorrect modeling of the variance-covariance matrix has on the results returned by a least-squares adjustment and on the resulting statistics.

The four station network described previously was used as a test network for various processing modes. Four hours of 0.5 Hz data were collected and processed using the GrafNavTM processor to yield reference coordinates for the stations.

Three 15 minute segments separated by one hour were removed from the data stream and then processed in a minimally constrained network adjustment using the reference coordinates of one receiver as fixed. Since the network is static, any changes in the reported receiver coordinates between the time periods can be considered as errors, and thus an indication of the accuracy of the processing mode.

The network adjustment was performed in four variations, namely :

- no modeling of the variance-covariance matrix and no accounting for the mathematical correlation between phases or $BCIB^T = s^2 I$ (NO MOD)
- no modeling of the variance-covariance matrix but accounting for the mathematical correlation between phases (MATH ONLY)
- modeling of the variance-covariance matrix as diagonal, using derived multipath and noise profiles, and accounting for the mathematical correlation between phases (VAR MOD)
- complete modeling of the variance-covariance matrix using the results of section 3 (FULL MOD)

Table 1 shows a comparison of results for the four tests for the three data segments. The actual RMS positional errors of all stations over the three test periods are shown, as is the predicted accuracy calculated from the variance-covariance matrix of estimated parameters returned by the least-squares adjustment. The standard deviations were calculated by scaling the variance-covariance matrix of parameters by the a posteriori variance factor for the adjustment. Particularly striking is the fact that the a posteriori variance factor for the adjustment comes

out very nearly equal to one when full modeling is performed. This is contrary to the typical over-optimistic results statistics reported by most commercial packages.

Table 1. Comparison of Processing Mode Results.

	METHOD (RMS / Predicted Acc.) all values in mm							
	NO MOD		MATH MOD		VAR MOD		FULL MOD	
N	3.3	0.4	3.1	0.4	2.6	0.4	3.1	3.4
E	2.2	0.3	1.3	0.3	1.5	0.5	1.3	2.5
H	3.9	0.6	3.4	0.6	3.6	0.8	4.5	6.4

Note however, that the actual results returned by the various methods only vary at the millimetre level, as indicated by the similar RMS levels observed. However, the estimated statistics vary by an order of magnitude. In fact, only by modelling the temporal correlation of multipath does the estimated accuracy approach the actual observed scatter in the reported positions.

6.0 Conclusions

A method of estimating the variance-covariance matrix of a set of double-differenced carrier phase observations has been presented, which depends on a theoretical modelling of noise and multipath errors, coupled with empirical determination of model parameters. It has been shown that improper modelling of the variance-covariance matrix has minimal effect on the actual positioning results on short-baselines, but that neglecting the temporal correlation of multipath causes the apostori parameter statistics to be seriously optimistic.

Noise and multipath were the two error sources considered, since all others essentially cancel in local-area networks. Noise was analysed through the use of a zero-baseline test, whereas a multipath model was developed through the analysis of baseline residuals, once the effects of noise were removed. In both cases, the variance model had the form of a zenith variance, multiplied by a mapping function, which itself was a function of antenna gain pattern. In addition, it was shown that the mapping functions for both noise and multipath could be derived from purely theoretical considerations, if information on the antenna gain pattern was available. If such information could not

be obtained, the mapping function could be empirically determined as well.

Finally, the temporal correlation of multipath was shown to follow an exponential decay with a correlation period of roughly 300 seconds. Further investigations are required to assess the change in the nature of the temporal correlation under various reflective environments, such as under wet and dry conditions.

7.0 References

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